
K3-SURFACES WITH SPECIAL SYMMETRY: AN EXAMPLE OF CLASSIFICATION BY MORI-REDUCTION

by

Kristina Frantzen and Alan Huckleberry

1. Introduction

An interesting class of K3-surfaces consists of those surfaces X equipped with an antisymplectic holomorphic automorphism $\sigma : X \rightarrow X$ of order two. Recently there has been substantial progress in understanding these manifolds, in particular as desingularized 2:1 ramified covers of log del Pezzo surfaces ([Na]), and analytic phenomena related to their moduli ([Y1, Y2]). Here we present an approach for studying such surfaces from the point of view of symmetry. This amounts to analyzing the action of a centralizer H of σ in the group of holomorphic symplectic automorphisms of the K3-surface X and the H -equivariant Mori-reduction of the quotient X/σ .

After presenting methods which apply in general, we turn to a special case where H is the nontrivial semidirect product $C_3 \ltimes C_7$ of cyclic groups of order three and seven. This arose naturally in our consideration of maximal groups of symplectic transformations, exemplifies the general approach but requires only minimal technical work. Furthermore, the results in this case shed new light on the classification of K3-surfaces with an action of the group $L_2(7)$, which has been studied extensively in [OZ]. (See [Z1] for a summary of other recent works in this direction.) Analogous classification results can be proved whenever H is of sufficiently high order or has sufficiently complicated group structure (see [FH]).

1.1. Notation. — The general problem can be formulated as follows. Let H be an abstract finite group, $A := C_2 = \langle \sigma \rangle$ and $G = A \times H$. An effective holomorphic G -action on a K3-surface X is defined by an injective homomorphism $\alpha : G \rightarrow \text{Aut}(X)$. It is assumed that $\alpha(\sigma)$ is antisymplectic and $\alpha(H) \subset \text{Aut}_{\text{sym}}(X)$ is a group of symplectic transformations of X . Abusing notation, if the context is clear, we do not differentiate between the abstract group or group elements and their α -images in $\text{Aut}(X)$, e.g., writing H for $\alpha(H)$. The set $\{(X, \alpha)\}$ of all G -actions on K3-surfaces is denoted by \mathcal{A} .

We wish to describe (up to natural equivalence) the K3-surfaces which are equipped with such actions. In precise terms, we regard actions (X_1, α_1) and (X_2, α_2) as being equivalent whenever there exists a biholomorphic map $\varphi : X_1 \rightarrow X_2$ and a group automorphism $\psi \in$

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$\text{Aut}(G)$ such that $\alpha_2(g)(\varphi(x)) = \varphi(\alpha_1(\psi(g))(x))$ for all $g \in G$ and $x \in X$. The goal is then to describe the quotient $\mathcal{M} = \mathcal{A}/\sim$ of all such actions by this equivalence relation.

1.2. Statement of results. — The fixed point set $\text{Fix}_X(\sigma)$ of an antisymplectic automorphism on a K3-surface X is a disjoint union of smooth curves. Unless this set is empty, the quotient $Y := X/\sigma$ is a smooth rational surface. Operating in the setting described above, we have the canonically induced H -action on Y . It is then possible to apply equivariant Mori-reduction to obtain an H -equivariant map $Y \rightarrow Y_{\min}$ to an H -minimal model which either has an ample anticanonical bundle or is an equivariant fiber space over \mathbb{P}_1 with general fiber \mathbb{P}_1 . If Z is a surface with $-K_Z$ ample and $K_Z^2 = d$, then $1 \leq d \leq 9$, and one refers to Z as a del Pezzo surface of degree d . Such a surface is either the blow up of \mathbb{P}_2 in $9 - d$ points in general position or $\mathbb{P}_1 \times \mathbb{P}_1$.

In §2, we study the exceptional curves of the equivariant Mori reduction $Y \rightarrow Y_{\min}$, in particular their intersection with the branch locus of the quotient map $\pi : X \rightarrow X/\sigma = Y$. We exemplify our method by classifying Y_{\min} and Y for H being the unique nontrivial semidirect product $C_3 \ltimes C_7$ defined by the action of C_3 on C_7 which is given by its embedding in $C_6 \cong \text{Aut}(C_7)$. Having done so, the classification problem for X reduces to a study of H -invariant sextics in \mathbb{P}_2 and we are able to describe the space \mathcal{M} by direct calculation (see Section 3.2).

Up to $\text{SL}_3(\mathbb{C})$ -conjugation there are two effective actions of H on \mathbb{P}_2 . These are equivalent in the above sense as they differ by an outer automorphism of H . A double cover of \mathbb{P}_2 branched along a smooth H -invariant curve of degree six is a K3-surface with an action of H . Those branch curves which define K3-surfaces with a symplectic action of H are defined by H -invariant polynomials. Choosing coordinates, the space $\mathbb{C}[z_0, z_1, z_2]_{(6)}^H$ of H -invariant polynomials of degree six is

$$V := \text{Span}\{z_0^2 z_1^2 z_2^2, z_0^5 z_1 + z_2^5 z_0 + z_1^5 z_2\}.$$

The family $\mathbb{P}(V)$ of curves defined by polynomials in V contains exactly four singular curves. These are the curve defined by $z_0^2 z_1^2 z_2^2$ and those defined by $3z_0^2 z_1^2 z_2^2 - \zeta^k(z_0^5 z_1 + z_2^5 z_0 + z_1^5 z_2)$, where ζ is a nontrivial cube root of unity, $k = 1, 2, 3$.

To define the equivalence relation which then yields a description of \mathcal{M} , we let Γ be the cyclic group of order three generated by the transformation $[z_0 : z_1 : z_2] \mapsto [z_0 : \zeta z_1 : \zeta^2 z_2]$, where ζ is as above. This group acts on the space $\mathbb{P}(V)$ of H -invariant curves and one shows that two such curves are equivalent by means of an automorphism of \mathbb{P}_2 if and only if they lie in the same Γ -orbit. In particular, the three irreducible singular H -invariant curves form a Γ -orbit.

The singular curve $C_{\text{sing}} \subset \mathbb{P}_2$ defined by $3z_0^2 z_1^2 z_2^2 - (z_0^5 z_1 + z_2^5 z_0 + z_1^5 z_2)$ has exactly seven singular points p_1, \dots, p_7 forming an H -orbit. Since they are in general position, the blow up of \mathbb{P}_2 in these points defines a del Pezzo surface Y_{Klein} of degree two with an action of H and is seen to be the double cover of \mathbb{P}_2 branched along Klein's quartic curve

$$C_{\text{Klein}} := \{z_0^3 + z_1^3 + z_2^3 = 0\}.$$

The proper transform B of C_{sing} in Y_{Klein} is a smooth H -invariant curve of genus three and coincides with the preimage of C_{Klein} in Y_{Klein} . The minimal resolution \tilde{X}_{sing} of the singular surface X_{sing} defined as the double cover of \mathbb{P}_2 branched along C_{sing} is a K3-surface with an action H . By construction, it is the double cover of Y_{Klein} branched along B . In particular, \tilde{X}_{sing} is the degree four cyclic cover of \mathbb{P}_2 branched along C_{Klein} and known as the Klein-Mukai-surface X_{KM} . The Klein-Mukai-surface is the unique K3-surface in the family \mathcal{M} such that $Y \not\cong \mathbb{P}_2$.

Thus, letting Σ be $\mathbb{P}(V)$ with the reducible curve $\{z_0^2 z_1^2 z_2^2 = 0\}$ removed, the parameter space \mathcal{M} is given by Σ/Γ and the description of K3-surfaces with G -actions can be formulated as follows.

Theorem 1. — *The K3-surfaces equipped with an action of $H = C_3 \ltimes C_7$ of holomorphic symplectic transformations which is centralized by an antisymplectic involution σ are parameterized by the space $\mathcal{M} = \Sigma/\Gamma$ of equivalence classes of sextic branch curves in \mathbb{P}_2 . The Klein-Mukai-surface occurs as the minimal desingularization of the double cover branched along the unique singular curve in \mathcal{M} .*

1.3. Application to the case $H = L_2(7)$. — In [Mu] Mukai gives a classification of abstract finite groups which can occur as groups of symplectic transformations of a K3-surface. Precisely, he presents a list of eleven finite groups such that a finite group occurs as a group of symplectic transformations on a K3-surface if and only if it occurs as a subgroup of one of the groups in this list. For every entry H in his list, Mukai gives an explicit example of a K3-surface with a symplectic H -action, but these examples are by no means exhaustive. It is therefore of interest to describe all surfaces where the groups from this list occur.

One first approach to understanding this situation is to consider finite groups G acting on K3-surfaces with normal subgroups H of symplectic transformations which are maximal in the above sense, i.e., which appear in Mukai's list. In other words, if we consider the G -action, $\omega \mapsto \chi(g) \cdot \omega$, on any symplectic form, then H is the kernel of the character χ and χ identifies G/H with some cyclic group C_k .

A particular example of a group on Mukai's list is $H = L_2(7)$. In this case, we may identify H with its group of inner automorphisms and consequently the map $G \rightarrow \text{Aut}(H)$ defined by conjugation induces a homomorphism

$$C_k \cong G/H \rightarrow \text{Out}(H) \cong C_2.$$

Therefore, except in the case where G is the only nontrivial semidirect product $G = C_2 \ltimes L_2(7)$, if $k \geq 2$, then H is centralized by a cyclic subgroup C_m of nonsymplectic transformations. In fact one can show that $m = 2, 4$ are the only possibilities (cf. [OZ]; for a proof using Morireduction see [F]). In both cases the existence of an antisymplectic involution σ centralizing H follows.

Remark 1.1. — Studying actions of a finite group G on K3-surfaces, one finds in many cases a cyclic group of nonsymplectic transformations as above which centralizes at least an interesting subgroup of the group of symplectic transformations in G . This is the general principle that led to our interest in this subject.

Now let us return to our original notation with $H = C_3 \ltimes C_7$, but here regarded as a subgroup of $L_2(7)$. If an involution σ centralizes $L_2(7)$, then it certainly centralizes H . Consequently, $\mathcal{M}(L_2(7))$ is contained in $\mathcal{M}(H)$. The following is proved by checking which elements of $\mathcal{M}(H)$ have the symmetry of the larger group $L_2(7)$.

Theorem 2. — *Among the K3-surfaces having a symplectic action of $C_3 \ltimes C_7$ centralized by an antisymplectic involution there are exactly two which are equipped with $L_2(7)$ -actions centralized by the same involution. These are X_{KM} and the surface defined as the 2:1 cover of \mathbb{P}_2 ramified over the curve $\text{Hess}(C_{\text{Klein}}) = \{z_0^5 z_1 + z_2^5 z_0 + z_1^5 z_2 - 5z_0^2 z_1^2 z_2^2 = 0\}$.*

Remark 1.2. — In general, the H -minimal model might not be that of a larger group. Here this occurs in the case of X_{KM} , where X_{KM}/σ is $L_2(7)$ -minimal but is not H -minimal.

The following result of Oguiso and Zhang ([OZ]) is a consequence of this theorem.

Corollary 1.3. — *If X is a K3-surface with an action of a finite group G containing $L_2(7)$ such that $|G/L_2(7)| = 4$, then X is the Klein-Mukai surface.*

Proof of Corollary 1.3. — Since $L_2(7)$ is simple and maximal in the above sense, it coincides with the group of symplectic transformations in G . In particular, $G/L_2(7) = C_4$ and a group $\langle \sigma \rangle$ of order two is contained in the kernel of $G \rightarrow \text{Aut}(L_2(7))$. Consequently we are in the setting of Theorem 2 where $\Lambda := G/\langle \sigma \rangle$ acts on $Y = X/\sigma$. If $X \neq X_{KM}$, then $Y = \mathbb{P}_2$. To complete the proof we must eliminate this possibility.

For this let τ be any element of Λ which is not in $L_2(7)$ and consider the conjugate $\tau H \tau^{-1}$. Since any two subgroups of order 21 in $L_2(7)$ are conjugate to each other by an element of $L_2(7)$, it follows that there exists $g \in L_2(7)$ with $(g\tau)H(g\tau)^{-1} = H$. Thus the normalizer $N(H)$ in Λ is a group of order 42 which also normalizes the commutator subgroup H' and therefore stabilizes its set F of fixed points.

Using the same coordinates $(z_0 : z_1 : z_2)$ of \mathbb{P}_2 as in Theorem 2 one directly checks that the only transformations in $\text{Stab}(F)$ which stabilize the branch curve $\text{Hess}(C_{\text{Klein}})$ are those in H itself. This contradiction shows that $Y \neq \mathbb{P}_2$ and therefore $X = X_{KM}$. \square

Remark 1.4. — The assumptions of the corollary may in fact be weakened. As we remarked above, one can show that $|G/L_2(7)| \geq 3$ implies $|G/L_2(7)| = 4$ (see [OZ] or [F]).

Let us conclude this introduction with a brief outline of this note. In §2 we explain the equivariant Mori-reduction and prove several general facts about the position of the Mori-fibers with respect to the images in Y of the σ -fixed point curves. These will be used in §3 to give the proofs of the theorems which are stated above. They will also play a fundamental role in [FH] where we prove analogous theorems for any group H which is either sufficiently large or sufficiently complicated.

2. General Methods

2.1. Quotients of K3-surfaces by antisymplectic involutions. — Continuing with the notation of Section 1.1, where H is an arbitrary finite group of symplectic automorphisms on a K3-surface X and σ is an antisymplectic involution which centralizes it, we begin by recalling that there are strong restrictions on the nature of $\text{Fix}(\sigma)$ ([N2]).

Proposition 2.1. — *If σ is an antisymplectic involution on a K3-surface X , then $\text{Fix}_X(\sigma)$ is one of the following types:*

$$1.) \text{Fix}_X(\sigma) = D_g \cup \bigcup_{i=1}^n R_i, \quad 2.) \text{Fix}_X(\sigma) = D_1^{(1)} \cup D_1^{(2)}, \quad 3.) \text{Fix}_X(\sigma) = \emptyset,$$

where D_g denotes a smooth curve of genus g and $\bigcup_{i=1}^n R_i$ is a possibly empty union of smooth disjoint rational curves.

To clarify the notation, this means in particular that if there are at least two elliptic curves in $\text{Fix}_X(\sigma)$, then the σ -fixed point set is the union of two elliptic curves, i.e., case 2.) occurs. It also should be remarked that the total number of curves in $\text{Fix}_X(\sigma)$ is at most 10 ([Z2]). Using this fact it would be possible to shorten one of our combinatorial arguments in the proof of Proposition 3.4, but instead we give a self-contained presentation.

Let $\pi : X \rightarrow X/\sigma = Y$ denote the quotient map. In the following, we assume that $\text{Fix}_X(\sigma)$ is nonempty. This implies that the quotient surface Y is a smooth rational surface with an action of H .

Remark 2.2. — Note that the centralizer of an automorphism stabilizes its set of fixed points. If h is a symplectic automorphism of order 7 on a K3-surface X , then $|\text{Fix}_X(h)| = 3$ ([N1]). In particular, if an automorphism σ of order two centralizes h , then $\text{Fix}_X(\sigma) \cap \text{Fix}_X(h) \neq \emptyset$.

2.2. The equivariant minimal model program for surfaces. — Mori's minimal model program may be adjusted to provide an equivariant version for projective varieties equipped with actions of finite groups (Example 2.18 in [KM]). Details in the case of interest in the present note, i.e., for smooth surfaces, are provided in [F]. Here we give a brief description of this *Mori-reduction*.

Let Y be a smooth projective surface with an action of a finite group H . Equivariant analogues of the cone and contraction theorems provide the following classification result.

Proposition 2.3. — *There exists a sequence of H -equivariant extremal contractions $Y \rightarrow Y_{(1)} \rightarrow \cdots \rightarrow Y_{\min}$ such that Y_{\min} satisfies one of the following conditions:*

1. $K_{Y_{\min}}$ is nef;
2. Y_{\min} is an H -equivariant conic bundle over a smooth curve, i.e., there exists an H -equivariant morphism $Y_{\min} \rightarrow C$ onto a smooth curve C such that the general fiber is a rational curve;
3. $-K_{Y_{\min}}$ is ample.

Each extremal contraction is the contraction of an H -orbit of disjoint (-1) -curves.

Definition 2.4. — The surface Y_{\min} is referred to as an *H -minimal model* of Y , and the map $Y \rightarrow Y_{\min}$ is called a *Mori-reduction*. A (connected) curve $E \subset Y$ is called *Mori-fiber* if it is contracted in some step of the Mori-reduction. The set of all Mori-fibers is denoted by $\mathcal{E}_{\text{Mori}}$.

In the present note we apply the equivariant minimal model program to the rational surface Y obtained as a quotient of the K3-surface X by an antisymplectic involution σ . Here Y is equipped with the action of the finite group H of holomorphic automorphisms which initially was acting on X and centralized by σ . An H -minimal model of Y can either be a del Pezzo surface or an equivariant conic bundle over \mathbb{P}_1 . We let n denote the number of rational curves in $\text{Fix}(\sigma)$.

Remark 2.5. — The Euler characteristic of the branched double cover $\pi : X \rightarrow Y$ is computed as $e(X) = 24 = 2e(Y) - \sum e(C)$, where the sum is taken over all connected components C of $\text{Fix}_X(\sigma)$. Hence

$$24 \geq 2e(Y) - 2n = 2e(Y_{\min}) + 2|\mathcal{E}_{\text{Mori}}| - 2n \geq 6 + 2|\mathcal{E}_{\text{Mori}}| - 2n$$

and the total number of Mori-fibers $|\mathcal{E}_{\text{Mori}}|$ is bounded by $n + 9$.

2.3. Branch curves and Mori-fibers. — Let $R := \text{Fix}_X(\sigma) \subset X$ denote the ramification locus of π and let $B = \pi(R) \subset Y$ be its branch locus. We denote by $\mathcal{E} \subset \mathcal{E}_{\text{Mori}}$ the set of all Mori-fibers which are not contained in the branch locus B .

Lemma 2.6. — *Let $E \in \mathcal{E}$ be a Mori-fiber such that $|E \cap B| \geq 2$ or $(E, B) \geq 3$. Then $\pi^{-1}(E)$ is a smooth rational curve in X .*

Proof. — Let $k < 0$ denote self-intersection number of E . The divisor $\pi^{-1}(E)$ has self-intersection $2k$. Assume that $\pi^{-1}(E)$ is reducible and let \tilde{E}_1, \tilde{E}_2 denote its irreducible components. They are rational and therefore have self-intersection number -2 . Write $2k = (\pi^{-1}(E))^2 = \tilde{E}_1^2 + \tilde{E}_2^2 + 2(\tilde{E}_1, \tilde{E}_2)$. Since \tilde{E}_1 and \tilde{E}_2 intersect at points in the preimage of $E \cap B$, we obtain $(\tilde{E}_1, \tilde{E}_2) \geq 2$, a contradiction. It follows that $\pi^{-1}(E)$ is irreducible. Consequently, $k = -1$ and $\pi^{-1}(E)$ is a smooth rational curve. \square

Remark 2.7. — Considering the preimage $\pi^{-1}(E) \subset X$ of a Mori-fiber $E \in \mathcal{E}$ it follows from adjunction on X that E is a (-1) -curve if and only if $E \cap B \neq \emptyset$; all Mori-fibers disjoint from B are (-2) -curves and all Mori-fibers $E \subset B$ are (-4) -curves. In particular, if $E_1, E_2 \in \mathcal{E}$ are two Mori-fibers which have nonempty intersection, then $E_1 \cap E_2$ is contained in the complement of B .

Proposition 2.8. — *Every Mori-fiber $E \in \mathcal{E}$ meets the branch locus B in at most two points. If E and B are tangent at p , then $E \cap B = \{p\}$ and $(E, B)_p = 2$.*

Proof. — Let $E \in \mathcal{E}$ and assume $|E \cap B| \geq 2$ or $(E, B) \geq 3$. Then by the lemma above, $\tilde{E} = \pi^{-1}(E)$ is a smooth rational curve in X . Since $\tilde{E} \not\subset \text{Fix}_X(\sigma)$, the involution σ has exactly two fixed points on \tilde{E} showing $|E \cap B| = 2$.

Let $N_{\tilde{E}}$ denote the normal bundle of \tilde{E} in X . We consider the induced action of σ on $N_{\tilde{E}}$ by a bundle automorphism. Using an equivariant tubular neighborhood theorem we may equivariantly identify a neighborhood of \tilde{E} in X with $N_{\tilde{E}}$ via a C^∞ -diffeomorphism. The σ -fixed point curves intersecting \tilde{E} map to curves of σ -fixed points in $N_{\tilde{E}}$ intersecting the zero-section and vice versa. Let D be a curve of σ -fixed point in $N_{\tilde{E}}$. If D is not a fiber of $N_{\tilde{E}}$, it follows that σ stabilizes all fibers intersecting D and the induced action of σ on the base must be trivial, a contradiction. It follows that the σ -fixed point curves correspond to fibers of $N_{\tilde{E}}$, and E and B meet transversally.

In particular, if E and B are tangent at p , then $|E \cap B| = 1$ and $(E, B) = 2$. \square

3. Fine classification

For the remainder of this paper $H := C_3 \ltimes C_7$, $A := C_2 = \langle \sigma \rangle$ and $G = A \times H$ is acting as in §1.1 on a K3-surface X . Here we prove the two theorems formulated in §1.

3.1. Representation as ramified cover. — Recall that the commutator subgroup $H' \cong C_7$ has exactly three fixed points in X ([N1]). Since H has no faithful 2-dimensional representation, it must act transitively on $\text{Fix}_X(H')$. Now σ stabilizes $\text{Fix}_X(H')$ as well and, as was mentioned above, has at least one fixed point there. However, since H acts transitively on $\text{Fix}_X(H')$ and centralizes σ , it follows that σ fixes $\text{Fix}_X(H')$ pointwise, i.e., $\text{Fix}_X(H') \subset \text{Fix}_X(\sigma)$ and in particular $\text{Fix}_Y(H')$ also consists of three points y_1, y_2, y_3 .

We now turn to an analysis of the branch curves of the covering $\pi : X \rightarrow X/\sigma = Y$.

Proposition 3.1. — *The set $\text{Fix}_Y(H')$ is contained in a unique (connected) branch curve B_0 . Its genus $g = g(B_0)$ is at least three.*

Proof. — Let B_i be the connected branch curve which contains y_i and observe that H acts transitively on the set $\mathcal{B} := \{B_1, B_2, B_3\}$. Since it is acting as C_3 , the set \mathcal{B} does not consist of two elements. If $|\mathcal{B}| = 3$, then from Proposition 2.1 it follows that at least two of the B_i must be rational. Since each B_i is stabilized by H' , it then follows that H' has at least two additional fixed points contradicting $|\text{Fix}_Y(H')| = 3$. Therefore \mathcal{B} consists of one curve which we denote by B_0 .

Now B_0 is H -invariant and, since it is acting as a group of symplectic transformations, H is acting effectively there. Furthermore, H' has exactly three fixed points in B_0 . Hence B_0 is not rational. There is indeed an elliptic curve with an effective H -action, but in such a case H' must act by translations, i.e., fixed point free. Thus B_0 is not elliptic. If $g(B_0) = 2$, then B_0 is hyperelliptic. But since the quotient of $B_0 \rightarrow \mathbb{P}^1$ by the hyperelliptic involution is $\text{Aut}(B_0)$ -equivariant and there is no effective action of H on \mathbb{P}^1 , this is also not possible. Consequently $g = g(B_0) \geq 3$ as claimed. \square

Let $\{B_1, \dots, B_n\}$ be the set of image curves which do not contain an H' -fixed point, i.e., $B_i \neq B_0$ for all i . Again using Proposition 2.1 we see that every B_i is rational. Thus, since H' has no fixed points in $\cup B_i$ for $i > 0$, we observe that H' is acting freely on $\{B_1, \dots, B_n\} = \mathcal{R}$.

Let us now first eliminate the possibility that an H -minimal model of Y is a \mathbb{P}_1 -fiber space.

Lemma 3.2. — *An H -minimal model of Y is a del Pezzo surface.*

Proof. — It is necessary to exclude the case of Y_{\min} being an H -equivariant conic bundle $Y_{\min} \rightarrow \mathbb{P}_1$. Since there is no effective action of H on \mathbb{P}_1 , it follows that the only proper normal subgroup H' of H acts trivially on the base. But the generic fibers are isomorphic to \mathbb{P}_1 and H' must have at least two fixed points in each such fiber, contrary to it having only three fixed points in Y . \square

Proposition 3.3. — *For every $B_i \in \mathcal{R}$ it follows that $|H.B_i| = 7$ and there are at most two such H -orbits in \mathcal{R} . In particular, $n \in \{0, 7, 14\}$.*

Proof. — Since $\text{Pic}(X)$ does not admit a negative-definite sublattice of rank 21 and H' stabilizes no curve in \mathcal{R} , the only possibility is $|H.B_i| = 7$ and $n \in \{0, 7, 14\}$. \square

The following is the main result of this section.

Proposition 3.4. — *There are no branch curves other than the one which contains $\text{Fix}_Y(H')$, i.e., $n = 0$ and $B = B_0$.*

Since the proof requires several combinatorial arguments, for the sake of clarity we separate it into a number of steps. Throughout we assume that $n \neq 0$ and at the end reach a contradiction. This assumption implies that \mathcal{R} consists of either one or two H -orbits consisting of seven curves and $n = 7, 14$, respectively. We make extensive use of the results in section 2.3 and the fact that an irreducible curve on a del Pezzo surface has self-intersection ≥ -1 . The desired contradiction will be that $|\mathcal{E}_{\text{Mori}}|$ is larger than the estimate guaranteed by the following observation.

Lemma 3.5. — *If $n = 7$, then $|\mathcal{E}_{\text{Mori}}|$ is at most 13. If $n = 14$, then $|\mathcal{E}_{\text{Mori}}|$ is at most 20.*

Proof. — Assume that $|\mathcal{E}_{\text{Mori}}|$ is larger than the claim. Then, using the Euler characteristic formula

$$13 - g(B_0) = e(Y_{\min}) + |\mathcal{E}_{\text{Mori}}| - n \quad (1)$$

along with $g(B_0) \geq 3$ and $e(Y_{\min}) \geq 3$, we see that $g = 3$, $n = 7$ implies $|\mathcal{E}_{\text{Mori}}| = 14$, $n = 14$ implies $|\mathcal{E}_{\text{Mori}}| = 21$, and $e(Y_{\min}) = 3$. Consequently, $Y_{\min} = \mathbb{P}_2$. The specified values of n and $|\mathcal{E}_{\text{Mori}}|$ guarantee $\mathcal{E}_{\text{Mori}} \cap \mathcal{R} = \mathcal{E}_{\text{Mori}} \setminus \mathcal{E} = \emptyset$. To see this, e.g., in the case where $n = 7$, note that if one curve B_i is contracted to a point by the reduction $Y \rightarrow Y_{\min}$, then all curves in \mathcal{R} are contracted and $|\mathcal{E}_{\text{Mori}} \setminus \mathcal{E}| = 7$. Since $B_i^2 = -4$, more than seven additional Mori-fibers are required in order to come to a step in the Mori-reduction where the curves B_i can be contracted. So all branch curves are mapped to curves in $Y_{\min} = \mathbb{P}_2$. However, using Remark 2.7 one sees that in this situation there is no configuration of Mori-fibers such that the images in $Y_{\min} = \mathbb{P}_2$ of every pair of branch curves have nonempty intersection. \square

Now let B_i be any branch curve in \mathcal{R} and $I_{B_i} := \text{Stab}_H(B_i)$. It follows that $I_{B_i} \cong C_3$.

Lemma 3.6. — *Every Mori-fiber which meets B_i is I_{B_i} -invariant and meets no other branch curve in the orbit $H.B_i$. Furthermore, B_i meets at most two Mori-fibers in \mathcal{E} .*

Proof. — Suppose there is an orbit $I_{B_i}.E_1 = \{E_1, E_2, E_3\}$ consisting of three different Mori-fibers which meet B_i . Let $S := \text{Stab}_H(E_1)$. It follows from Lemma 3.5 above that S is nontrivial. If S does not stabilize B_i , then $|S.B_i| \geq 3$, contrary to $|E_1 \cap B| \leq 2$ (cf. Proposition 2.8). Thus $S = I_{B_i}$ and $I_{B_i}.E_1 = \{E_1\}$. If E_1 meets any other branch curve in $H.B_i$, then it meets at least three others, again contrary to $|E_1 \cap B| \leq 2$. Finally, if B_i meets more than two Mori-fibers in \mathcal{E} , then at least one Mori-fiber E meets B_i outside $\text{Fix}_{B_i}(I_{B_i})$. Since E is I_{B_i} -invariant, it follows that $E \cap B$ consists of more than three points, a contradiction. \square

Lemma 3.7. — *Every $B_i \in \mathcal{R}$ meets exactly one Mori-fiber in \mathcal{E} .*

Proof. — First note that a (-4) -curve in Y cannot be mapped biholomorphically to the del Pezzo surface Y_{\min} and therefore every curve $B_i \in \mathcal{R}$ meets at least one Mori-fiber. Assume some B_i meets at least three Mori-fibers. Since none of these meets another branch curve in the orbit $H.B_i$ by Lemma 3.6, we obtain $|\mathcal{E}_{\text{Mori}}| \geq 21$, contrary to Lemma 3.5. If $n = 7$, then the same argument proves the desired result. It remains to consider the case $n = 14$. If B_i meets two Mori-fibers E_1, E_2 and both intersections are transversal, then, since the image of B_i in the del Pezzo surface Y_{\min} cannot be a (-2) -curve, at least one of these two Mori-fibers meets a third one E_3 . Since $E_1^2 = E_2^2 = -1$ and $E_3^2 \leq -2$ (cf. Remark 2.7) this Mori-fiber E_3 is not among $H.E_1$ or $H.E_2$ and by Lemma 3.6 the full configuration consists of at least 21 Mori-fibers. If B_i meets two Mori-fibers E_1, E_2 and E_1 is tangent to B_i , then Proposition 2.8 ensures that E_1 meets no other curve in \mathcal{R} . Even if E_2 meets a branch curve B_k in the other H -orbit $\mathcal{R} \setminus (H.B_i)$ additional Mori-fibers meeting $H.B_k$ are required and again the total number of Mori-fibers exceeds 21. \square

We have now reduced to the situation where every branch curve $B_i \in \mathcal{R}$ meets exactly one Mori-fiber $E \in \mathcal{E}$. This final case requires a closer look at the intersection diagram of branch curves and Mori-fibers: each $B_i \in \mathcal{R}$ fulfills one of the following possibilities:

1. $E \cap B_i = \{p_1, p_2\}$ or
2. $E \cap B_i = \{p\}$ and $(E, B_i)_p = 2$ or
3. $E \cap B_i = \{p\}$ and $(E, B_i)_p = 1$.

Proof of Proposition 3.4. — In cases 1.) and 2.), by Proposition 2.8 the only branch curve which is met by E is B_i itself. The blowing down of E transforms B_i into a singular curve of self-intersection zero. Since a del Pezzo surface does not admit a curve of this type, there exists a Mori-fiber $E_1 \in \mathcal{E}$ with $E_1 \cap E \neq \emptyset$. By Remark 2.7 the Mori-fiber E_1 does not meet the branch locus B and $E_1^2 = -2$. Furthermore, E_1 meets no (-1) -curve among the Mori-fibers except E . Thus we have found 14 Mori-fibers along the orbit $H.B_i$. If $n = 7$, this yields the desired contradiction. If $n = 14$, then none of these 14 Mori-fibers meets a branch curve in the other H -orbit, which must therefore contribute at least 7 additional Mori-fibers. The total number of Mori-fibers in this configuration violates Lemma 3.5.

It remains to consider the situation where all intersections of $B_i \in \mathcal{R}$ with Mori-fibers are as in case 3.). As above, we deduce the existence of a Mori-fiber E_1 which meets E in exactly one point. If $n = 7$, we have reached the desired contradiction. Hence, we may suppose that $n = 14$.

Let us check that without loss of generality we can assume that $E_1 \in \mathcal{E}$, i.e., $E_1 \notin \mathcal{R}$. If $E_1 \in \mathcal{R}$, i.e., $E_1 = B_k$ for some $B_k \in \mathcal{R} \setminus (H.B_i)$, then blowing down E transforms $B_i \cup B_k$ into a pair of intersecting (-3) -curves. It follows that there exists a Mori-fiber $E_2 \in \mathcal{E}$ with $E_2 \cap E \neq \emptyset$ which we then pick instead of E_1 .

So let $E_1 \in \mathcal{E} = \mathcal{E}_{\text{Mori}} \setminus \mathcal{R}$. In particular, E_1 has self-intersection -2 , meets no branch curve, no (-1) -curve among the Mori-fibers except E , and meets E in exactly one point. It follows that E_1 must be I_{B_i} -invariant, since otherwise we find at least 28 Mori-fibers along $H.B_i$.

If E meets a branch curve B_k in the other H -orbit, then B_k must also be I_{B_i} -invariant, as otherwise E would meet at least four branch curves. We obtain a contradiction since the I_{B_i} -action on E may not fix the three points of intersection of E with B_i , B_k and E_1 . Consequently, E cannot meet a branch curve in the other H -orbit, which therefore contributes at least seven Mori-fibers, and $|\mathcal{E}_{\text{Mori}}| \geq 21$. \square

If $X/\sigma = Y$ is not H -minimal, then it is characterized by the following observation.

Proposition 3.8. — *The surface Y is either H -minimal or the blow up of \mathbb{P}_2 in seven singularities of an irreducible H -invariant sextic..*

Proof. — Since $n = 0$, the Euler characteristic formula (1) yields $|\mathcal{E}| \leq 7$. The fact that H acts on \mathcal{E} then implies that $|\mathcal{E}| \in \{0, 3, 6, 7\}$. If $|\mathcal{E}| \in \{3, 6\}$, then H' stabilizes every $E \in \mathcal{E}$, and consequently it has more than three fixed points, a contradiction. Thus we must only consider the case $|\mathcal{E}| = 7$.

Since \mathcal{E} is an H -orbit, it follows that every $E \in \mathcal{E}$ has self-intersection -1 and therefore has nonempty intersection with B_0 by Remark 2.7.

The Euler characteristic formula again implies that $g(B_0) = 3$ and $Y_{\min} = \mathbb{P}_2$ and adjunction in X shows that $B_0.B_0 = 8$ in Y . The fact that B_0 has nonempty intersection with seven different Mori-fibers implies that its image C in Y_{\min} has self-intersection either $15 = 8 + 7$ or $36 = 8 + 4 \cdot 7$. Since the first is impossible it follows that $(E, B_0) = 2$ for all $E \in \mathcal{E}$ and the H -invariant irreducible sextic C has seven singular points corresponding to the images of E in \mathbb{P}_2 . \square

Corollary 3.9. — *If Y is not H -minimal, then X is the minimal desingularization of a double cover of \mathbb{P}_2 branched along an irreducible H -invariant sextic with seven singular points.*

We conclude this subsection with a classification of possible H -minimal models of Y .

Proposition 3.10. — *The surface Y_{\min} is either a del Pezzo surface of degree two or \mathbb{P}_2 .*

Proof. — If $Y_{\min} = \mathbb{P}_1 \times \mathbb{P}_1$, then, since there are no nontrivial homomorphisms of H to C_2 , it follows that H is contained in the connected component of $\text{Aut}(Y_{\min})$. But this is also not possible, because there are no injective homomorphisms of H into $\text{PSL}_2(\mathbb{C})$. Thus $Y_{\min} = Y_d$ is a del Pezzo surface of degree $d = 1, \dots, 9$ which is a blowup of \mathbb{P}_2 in $9 - d$ points.

It is immediate that $d \neq 1$, because the anticanonical map of such a surface has exactly one base point. Since this would have to be H -fixed and H has no faithful 2-dimensional representations, this case does not occur and we must only eliminate $d = 3, \dots, 8$. However, in these cases the sets \mathcal{K} of (-1) -curves consist, respectively, of 27, 16, 10, 6, 2, 1 elements (see, e.g., [Ma]). The H -orbits in \mathcal{K} consist of either 1, 3, 7 or 21 curves and clearly either 1 or 3 must occur in every case. If H had a fixed curve in \mathcal{K} , then we could blow it down to obtain a 2-dimensional representation of H which does not exist. If H had an orbit consisting of three curves, H' would stabilize each of the curves in that orbit. Thus H' would have at least six fixed points in Y_{\min} and in Y . This contradicts the fact that $|\text{Fix}_Y(H')| = 3$. \square

3.2. Computation of invariants and equivalence. — The results of the previous section reduce the problem of parameterizing the equivalence classes of K3-surfaces equipped with actions $\alpha \in \mathcal{A}$ to studying equivariant 2:1-covers of surfaces $Y = X/\sigma$ branched along a single curve of genus ≥ 3 . The surface Y is either \mathbb{P}_2 , the blow up of \mathbb{P}_2 in seven singularities of an

irreducible H -invariant sextic, or a del Pezzo surfaces of degree two. Let us begin with a study of the first two cases where $Y_{\min} = \mathbb{P}_2$

The case where $Y_{\min} = \mathbb{P}_2$. — Here we will show that if $Y_{\min} = \mathbb{P}_2$, then X is either a double cover of \mathbb{P}_2 branched along a smooth H -invariant sextic or the minimal desingularization of a double cover of \mathbb{P}_2 branched along a unique sextic with seven singular points which is described below.

An effective action of H on \mathbb{P}_2 is given by an injective homomorphisms $H \rightarrow \mathrm{PGL}_3(\mathbb{C})$. Due to its group structure, the only central extension of H by C_3 is trivial. Thus, we may regard H as a subgroup of $\mathrm{SL}_3(\mathbb{C})$ acting by one of its two irreducible 3-dimensional representations. Since these representations differ by a group automorphism and the corresponding actions on \mathbb{P}_2 are therefore equivalent, we may assume that we are only dealing with the following case: in appropriately chosen coordinates a generator of H' acts by $[z_0 : z_1 : z_2] \mapsto [\lambda z_0, \lambda^2 z_1, \lambda^4 z_2]$, where $\lambda = \exp \frac{2\pi i}{7}$ and a generator of C_3 acts by the cyclic permutation τ which is defined by $[z_0 : z_1 : z_2] \mapsto [z_2 : z_0 : z_1]$.

Any homogeneous polynomial defining an invariant curve must be an H -eigenfunction with H' acting with eigenvalue one. It is a simple matter to compute the H' -invariant monomials:

$$\mathbb{C}[z_0, z_1, z_2]_{(6)}^{H'} = \mathrm{Span}\{z_0^2 z_1^2 z_2^2, z_0^5 z_1, z_2^5 z_0, z_1^5 z_2\}.$$

Letting $P_1 = z_0^2 z_1^2 z_2^2$ and $P_2 = z_0^5 z_1 + z_2^5 z_0 + z_1^5 z_2$, it follows that

$$\mathbb{C}[z_0, z_1, z_2]_{(6)}^H = \mathrm{Span}\{P_1, P_2\} =: V.$$

There are two 1-dimensional H -eigenspaces, i.e., those spanned by $z_0^5 z_1 + \zeta z_2^5 z_0 + \zeta^2 z_1^5 z_2$ for $\zeta^3 = 1$ but $\zeta \neq 1$. By direct computation one checks that the curves defined by these polynomials are smooth and that in both cases all τ fixed points in \mathbb{P}_2 lie on them. Thus, τ has only three fixed points on the K3-surface X obtained as a double cover. But τ generates a copy of C_3 which, if it would act by symplectic transformations, would have six fixed points in X ([N1]). Consequently, H does not lift to an action by symplectic transformations on the K3-surfaces defined by these curves. Hence it is enough to consider ramified covers $X \rightarrow Y = \mathbb{P}_2$, where the polynomials defining the branch curves are invariant.

Therefore the curves under discussion are parameterized by the 1-dimensional projective space $\mathbb{P}(V)$. Our first step is to determine which polynomials $P_{\alpha, \beta} = \alpha P_1 + \beta P_2$ define singular curves $C = \{P = 0\}$. Clearly P_1 is such an example. The location of the τ -fixed points is key for the determination of the other singular curves. Since $\mathrm{Fix}(\tau)$ consists of the three points $[1 : \zeta : \zeta^2]$, where $\zeta^3 = 1$, the curves which contain τ -fixed points are defined by condition $\alpha + 3\zeta\beta = 0$. Let C_{P_1} be the curve defined by P_1 and C_ζ be that defined by $P_{\alpha, \beta}$ which satisfies the above condition. A direct computation shows that the C_ζ are singular at the corresponding τ -fixed point. So we let Σ_{reg} be the complement of this set of four curves in $\mathbb{P}(V)$ and note the following.

Proposition 3.11. — *A curve $C \in \mathbb{P}(V)$ is smooth if and only if it is in Σ_{reg} . Furthermore, for every such C the H -action on \mathbb{P}_2 lifts to a group of symplectic transformations on the K3-surface X which is defined as the 2:1 cover of \mathbb{P}_2 ramified over C . The H -action is centralized by the antisymplectic involution which defines the covering $X \rightarrow \mathbb{P}_2$.*

Proof. — First we show that every curve $C \in \Sigma_{\mathrm{reg}}$ is smooth. For this observe that, since τ has no fixed points in C and every subgroup of order three in H is conjugate to that generated by τ , $|H.p| = 3, 21$ for every $p \in C$. Now the only subgroup of order seven in H is the commutator group H' . So the only possible H -orbits having three elements are the orbits of the H' -fixed

points. So we pick one such fixed point p and directly check that every $C \in \Sigma_{\text{reg}}$ is smooth at p . Thus if C is singular at some point q , then it is singular at each of the 21 points in $H.q$. By considering the H -action on the space of irreducible components of C , one checks that C is irreducible, and therefore the genus formula can be applied to show that C has at most 10 singularities. Hence C is smooth.

Now the preimage of H in $\text{Aut}(X)$ is a central extension of H by C_2 . This splits as $H \times C_2$, where the second factor is generated by the 2:1 covering map σ which acting antisymplectically. Since the commutator group H' automatically acts by symplectic transformations, we must only check that the lift of the cyclic permutation τ , $[z_0 : z_1 : z_2] \mapsto [z_2 : z_0 : z_1]$, acts symplectically. The local linearization of τ at the fixed point $[1 : 1 : 1] \in \mathbb{P}_2$ is in $\text{SL}_2(\mathbb{C})$. Since this fixed point is in the complement of C , its local linearization at a corresponding fixed point in X is also in $\text{SL}_2(\mathbb{C})$ and consequently it is acting symplectically. \square

Let us now turn to a description of $\mathcal{M}_{\mathbb{P}_2} := \mathcal{A}_{\mathbb{P}_2} / \sim$. Here the index \mathbb{P}_2 indicates that we have restricted to the case where $Y = Y_{\min} = \mathbb{P}_2$. Using the covering $X \rightarrow \mathbb{P}_2$ the set $\mathcal{A}_{\mathbb{P}_2}$ becomes the set of curves $\{h(C) \mid C \in \Sigma_{\text{reg}}, h \in \text{PGL}_3(\mathbb{C})\}$. Equivalence is also defined by the action of $\text{PGL}_3(\mathbb{C})$ so that $\mathcal{M}_{\mathbb{P}_2}$ can be identified with $\Sigma_{\text{reg}} / \sim$, where $C, \tilde{C} \in \Sigma_{\text{reg}}$ are equivalent if and only if there exists $h \in \text{PGL}_3(\mathbb{C})$ with $h(C) = \tilde{C}$. Of course the normalizer $N := N(H)$ in $\text{PGL}_3(\mathbb{C})$ stabilizes $\mathbb{P}(V)$, because $\mathbb{P}(V)$ can be viewed as the set of H -fixed points in $\mathbb{P}(\mathbb{C}[z_0, z_1, z_2]_{(6)})$.

The group N is the product $\Gamma \times H$, where Γ is the cyclic group of order three generated by the transformation $[z_0 : z_1 : z_2] \mapsto [z_0, \zeta z_1, \zeta^2 z_2]$, where as above $\zeta^3 = 1$. Note that Γ acts transitively on the set of three singular curves $\{C_{\tilde{\zeta}}\}$ and stabilizes the curve C_{P_1} . It also stabilizes the curve C_{P_2} defined by the polynomial P_2 .

Proposition 3.12. — *The equivalence relation on Σ_{reg} is that defined by the Γ -action, i.e.,*

$$\mathcal{M}_{\mathbb{P}_2} = \Sigma_{\text{reg}} / \Gamma.$$

Proof. — Let $C \in \Sigma_{\text{reg}}$ and for $T \in \text{SL}_3(\mathbb{C})$ assume that $T(C) \in \Sigma_{\text{reg}}$. Consider the group span S of THT^{-1} and H . We have shown that H acts as a group of symplectic transformations on every K3-surface defined by a curve in Σ_{reg} . This was proved by considering the linearization at the τ -fixed points and their location. Therefore the same argument shows that THT^{-1} also acts as a group of symplectic transformations on the K3-surface associated to $T(C)$. Thus S is acting as a group of symplectic transformations on this K3-surface.

Now if $S = H$, then T normalizes H and, modulo H , is contained in Γ . To handle the case where $S \neq H$, note that $L_2(7)$ is the only group in Mukai's list which contains a copy of H . Therefore if $S \neq H$, then, since S can be realized as a subgroup of $L_2(7)$ and H is realized as a maximal subgroup, it follows that $S = L_2(7)$. But there is only one curve in the family $\mathcal{M}_{\mathbb{P}_2}$ which is $L_2(7)$ -invariant, namely $\text{Hess}(C_{\text{Klein}})$. Thus $C = T(C) = \text{Hess}(C_{\text{Klein}})$ and the desired result follows. \square

Remarks. — (1) If two K3-surfaces X_1 and X_2 are biholomorphically equivalent by a map $\varphi : X_1 \rightarrow X_2$ and X_1 comes equipped with a G -action so that it is in our family $\mathcal{A}_{\mathbb{P}_2}$, then using φ we equip X_2 with an equivalent G -action. Thus we may regard X_1 and X_2 as equivalent points in Σ_{reg} . As a result we observe that no two K3-surfaces parameterized by $\mathcal{M}_{\mathbb{P}_2}$ are biholomorphically equivalent. (2) The group Γ stabilizes the curve C_{P_2} and therefore acts together with H on the associated K3-surface X . Now the only group in Mukai's list which contains a copy of H is $L_2(7)$ and as a subgroup H is maximal. Therefore Γ is acting nonsymplectically on X .

Our study of H -invariant sextic curves in \mathbb{P}_2 has revealed the existence of precisely three singular irreducible examples C_ζ with $\zeta^3 = 1$. These are identified by the action of Γ . If $Y = X/\sigma$ is not H -minimal, it follows that up to equivariant equivalence the K3-surface X is the minimal desingularization of the double cover X_{sing} of \mathbb{P}_2 branched along $C_{\zeta=1} = C_{\text{sing}}$, i.e., the K3-surface arising in Corollary 3.9 is precisely this surface. In particular, Y is the blow up of \mathbb{P}_2 in the seven singular points of C_{sing} , which are given as the H' -orbit of $[1 : 1 : 1] \in \mathbb{P}_2$.

Proposition 3.13. — *If Y is not minimal, then it is the del Pezzo surface of degree two which arises by blowing up the seven singular points p_1, \dots, p_7 on the curve C_{sing} in \mathbb{P}_2 . The corresponding map $Y \rightarrow \mathbb{P}_2$ is H -equivariant and therefore a Mori-reduction of Y . The branch curve B_0 of $\pi : X \rightarrow Y$ is the proper transform of C_{sing} in Y .*

Proof. — We need to show that the points $\{p_1, \dots, p_7\} = H' \cdot [1 : 1 : 1]$ are in general position, i.e., no three lie on one line and no six lie on one conic. It follows from direct computation that no three points in $H' \cdot [1 : 1 : 1]$ lie on one line. If p_1, \dots, p_6 lie on a conic Q , then $h \cdot p_1, \dots, h \cdot p_6$ lie on $h \cdot Q$ for every $h \in H$. Since $\{p_1, \dots, p_7\}$ is an H -invariant set, the conics Q and $h \cdot Q$ intersect in at least five points and therefore coincide. It follows that Q is an invariant conic meeting C_{sing} at its seven singularities and $(Q, C_{\text{sing}}) = 14$ implies $Q \subset C_{\text{sing}}$, a contradiction. \square

The role of Klein's quartic. — Here we show that the del Pezzo surface which arises as the blow up of the seven singular points on C_{sing} can also be regarded as the 2:1 cover of \mathbb{P}_2 ramified over Klein's quartic curve. For this recall that the anticanonical map of a del Pezzo surface Y of degree two realizes it as a 2:1-cover $Y \rightarrow \mathbb{P}_2$ which is ramified over a smooth curve C of degree four. This map is equivariant with respect to the full automorphism group $\text{Aut}(Y)$. Conversely, if C is any smooth curve of degree four in \mathbb{P}_2 , and Y is defined as the 2:1-cover of \mathbb{P}_2 which is ramified along C , then Y is a del Pezzo surface of degree two. Furthermore, the elements of the stabilizer in $\text{Aut}(\mathbb{P}_2)$ of the curve C are exactly those transformations which lift to automorphisms of Y .

Now assume as in our case that Y comes equipped with an H -action so that H can be regarded as a subgroup of $\text{PGL}_3(\mathbb{C})$ which stabilizes a smooth quartic curve $C \subset \mathbb{P}_2$. In order to determine the possibilities for C , we choose coordinates so that H is acting as above. It follows that

$$\mathbb{C}[z_0 : z_1 : z_2]_{(4)}^{H'} = \text{Span}\{z_0^3 z_2, z_1^3 z_0, z_2^3 z_1\}.$$

This is a direct sum of H -eigenspaces with the eigenspace of the eigenvalue ζ being spanned by the polynomial $Q_\zeta := z_0^3 z_2 + \zeta z_2^3 z_1 + \zeta^2 z_1^3 z_0$ with ζ arbitrary such that $\zeta^3 = 1$.

Now consider the cyclic group $\Gamma \subset \text{GL}_3(\mathbb{C})$ which is generated by the transformation γ , $(z_0, z_1, z_2) \mapsto (z_0, \zeta z_1, \zeta^2 z_2)$. The group Γ acts transitively on the H -eigenspaces spanned by the Q_ζ . Consequently, up to equivariant equivalence, we may assume that $Y \rightarrow \mathbb{P}_2$ is ramified over Klein's curve C_{Klein} which is defined by Q_1 . We therefore have the following observation.

Proposition 3.14. — *A del Pezzo surface of degree two with an action of H is equivariantly isomorphic to the double cover Y_{Klein} of \mathbb{P}_2 branched along Klein's quartic curve with the action of H on \mathbb{P}_2 defined as above.*

In particular, the Mori-reduction of any H -action on Y_{Klein} is equivalent to the one defined above and in summary we have the following result.

Proposition 3.15. — *If X is a K3-surface equipped with an H -action which centralizes an antisymplectic involution σ , then $Y_{\min} = \mathbb{P}_2$. In all but one case $X/\sigma = Y = Y_{\min}$. In the exceptional case $Y = Y_{\text{Klein}}$, the Mori-reduction $Y \rightarrow Y_{\min}$ is the blow down of seven (-1) -curves to the singular points of C_{sing} and the branch set B_0 of $X \rightarrow Y$ is the proper transform of C_{sing} in Y .*

Proof. — It remains to prove that B_0 is the proper transform of C_{sing} in Y . For this suppose that the branch curve of $X \rightarrow Y$ were some other curve \tilde{B}_0 in the linear system of $-2K_Y$. For $I := \tilde{B}_0 \cap B_0$ we note that $|I| \leq 8$. But since H has no fixed points in B_0 , it follows that $|I| = 3$ and that I is an H -orbit. Thus the intersection multiplicities of $\tilde{B}_0 \cap B_0$ are the same along I , contrary to the fact that 3 does not divide 8. \square

Completion of the proof of Theorem 1. — To complete the proof of Theorem 1 we must first show that if H is acting on \mathbb{P}_2 as above, then it lifts to a group acting on the K3-surface X which is defined as the 2:1 cover of the del Pezzo surface Y_{Klein} which in turn is defined as the cover of \mathbb{P}_2 ramified over C_{Klein} . Of course we mean that $X \rightarrow Y_{\text{Klein}}$ is ramified over the preimage B_0 of C_{Klein} . Since H stabilizes C_{Klein} and does not admit nontrivial central extensions of degree two, it lifts to a subgroup of $\text{Aut}(Y_{\text{Klein}})$. By the same argument H lifts to a subgroup of $\text{Aut}(X)$.

Secondly, the covering involution $Y_{\text{Klein}} \rightarrow \mathbb{P}_2$, lifts to a holomorphic transformation of X . On X we also consider the involution defining $X \rightarrow Y_{\text{Klein}}$. Together these transformations generate a group of order four, every element of which has a positive-dimensional fixed point set. Therefore this group acts as C_4 by a character on any choice of the symplectic form of X . Thus the full preimage of H in $\text{Aut}(X)$ splits uniquely as a product $H \times C_4$. Since the commutator group H' automatically acts by symplectic transformations, we must only check that the lift of the cyclic permutation $\tau, [z_0 : z_1 : z_2] \mapsto [z_2 : z_0 : z_1]$, acts symplectically. As before, this follows from a linearization argument at a τ -fixed point not in C_{Klein} .

Thus we observe that in this situation up to equivalence there is a unique action of H by symplectic transformations on a unique K3-surface X_{KM} , and this is centralized by a cyclic group of order four which acts faithfully as C_4 by a multiplicative character on any choice of the symplectic form.

It follows that $\mathcal{M} \setminus \mathcal{M}_{\mathbb{P}_2} = \{X_{KM}\}$, i.e., the Klein-Mukai-surface is the only surface in the family \mathcal{M} for which $Y \not\cong \mathbb{P}_2$. If we define Σ as the complement of C_{P_1} in $\mathbb{P}(V)$. Then $\Sigma = \Sigma_{\text{reg}} \cup \{C_{\zeta} \mid \zeta^3 = 1\}$ and

$$\mathcal{M} = \Sigma / \Gamma.$$

The above discussion completes the proof of Theorem 1. It remains to determine which K3-surfaces in the family \mathcal{M} have $L_2(7)$ -symmetry.

3.3. The $L_2(7)$ -action. — First observe that an action of $L_2(7)$ on \mathbb{P}_2 is necessarily given by one of its two 3-dimensional irreducible representations, which differ by an outer automorphism of the group. We may therefore consider one particular representation such that the subgroup H is represented as above and check that the curve $\text{Hess}(C_{\text{Klein}}) \in \Sigma_{\text{reg}}$ is $L_2(7)$ -invariant. It is in fact the unique curve in Σ_{reg} with this property. This is a well-known result from the invariant theory of the group $L_2(7)$ but can also be seen as follows: suppose there were two distinct invariant smooth sextic curves C_s, C'_s . The maximal possible isotropy of $L_2(7)$ is the cyclic group of order 7 so that each $L_2(7)$ -orbit on C_s has at least 21 elements. It follows that there is no configuration of C_s and C'_s which fulfills $C_s \cdot C'_s = 36$, a contradiction.

We have hereby singled out a unique K3-surface with $L_2(7)$ -symmetry in the family $\mathcal{M}_{\mathbb{P}_2}$. Since $L_2(7)$ is a simple group and in particular is equal to its commutator, its action on this surface is clearly symplectic and by construction is centralized by the antisymplectic covering involution.

To complete the proof of Theorem 2, note that the curve $C_{\text{Klein}} \subset \mathbb{P}_2$ is $L_2(7)$ -invariant with respect to the representation discussed above. We see that $L_2(7)$ lifts to a subgroup of $\text{Aut}(X_{KM})$. Hence, analogous to the case of H , the preimage of $L_2(7)$ in $\text{Aut}(X_{KM})$ is

$L_2(7) \times C_4$, where $L_2(7)$ is acting symplectically and C_4 is acting by holomorphic nonsymplectic transformations. The generator σ of the subgroup isomorphic to C_2 in C_4 is the anti-symplectic involution which centralizes $L_2(7)$ in the above discussion.

In conclusion we reiterate that both of the K3-surfaces which have an $L_2(7)$ -action centralized by an involution appear in the family parameterized by \mathcal{M} . For the K3-surface constructed as the 2:1 cover of \mathbb{P}_2 ramified along $\text{Hess}(C_{\text{Klein}})$ we clearly have $Y = Y_{\min} = \mathbb{P}_2$ for both H and $L_2(7)$. For X_{KM} the quotient $Y = Y_{\text{Klein}}$ is an $L_2(7)$ -minimal model, whereas \mathbb{P}_2 is the minimal model with respect to the action of H .

References

- [F] Kristina Frantzen, *On K3-surfaces with finite symmetry groups of mixed type*, Dissertation, Ruhr-Universität Bochum, expected 2008.
- [FH] Kristina Frantzen and Alan Huckleberry, *On K3-surfaces with symplectic symmetry centralized by an involution*, in preparation.
- [KM] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998.
- [Ma] Yuri I. Manin, *Cubic forms: algebra, geometry, arithmetic*, North-Holland Publishing Co., Amsterdam, 1974, North-Holland Mathematical Library, Vol. 4.
- [Mu] Shigeru Mukai, *Finite groups of automorphisms of K3 surfaces and the Mathieu group*, Invent. Math. **94** (1988), no. 1, 183–221.
- [Na] Noboru Nakayama, *Classification of Log del Pezzo surfaces of Index Two*, J. Math. Sci. Univ. Tokyo **14** (2007), 293–498.
- [N1] Viacheslav V. Nikulin, *Finite groups of automorphisms of Kählerian K3 surfaces*, Trans. Moscow Math. Soc. **38** (1979), no. 2, 71–135.
- [N2] Viacheslav V. Nikulin, *Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. Algebrogeometric applications*, J. Soviet Math. **22** (1983), 1401–1476.
- [OZ] Keiji Oguiso and De-Qi Zhang, *The simple group of order 168 and K3 surfaces*, Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, 165–184, ArXiv 0011259.
- [Y1] Ken-Ichi Yoshikawa, *K3 surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space*, Invent. Math. **156** (2004), no. 1, 53–117.
- [Y2] Ken-Ichi Yoshikawa, *K3 surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space II: a structure theorem*, University of Tokyo, Graduate School of Mathematical Sciences, preprint, 2007.
- [Z1] De-Qi Zhang, *Automorphisms of K3 surfaces*, Proceedings of the International Conference on Complex Geometry and Related Fields, AMS/IP Stud. Adv. Math. **39** (2007), 379–392, ArXiv 0506612.
- [Z2] De-Qi Zhang, *Quotients of K3 surfaces modulo involutions*, Japan. J. Math.(N.S.), **24** (1998). no. 2, 335–366, ArXiv 9905193.

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KRISTINA FRANTZEN AND ALAN HUCKLEBERRY, Institut und Fakultät für Mathematik, Ruhr-Universität Bochum, Germany, kristina.frantzen@ruhr-uni-bochum.de, ahuck@cplx.ruhr-uni-bochum.de